Indian Statistical Institute
First Semester Exam, 2006-2007
M.Math.II Year
Operator Theory

Time: 3 hrs   Date:06-12-06   Marks : $10 \times 6 = 60$

Instructor: T S S R K Rao

Give full details wherever necessary.

1. Let X be a separable Banach space and let $\{x_n^*\}_{n\geq 1}$ be a weak*-dense sequence in the dual unit ball. Define $\Phi : X \to \ell^\infty$ by $\Phi(x) = \{x_n^*(x)\}_{n\geq 1}$. Show that the range of Φ^* is weak*-closed.

2. Let X and Y be Banach spaces such that the space of operators $\mathcal{L}(X, Y)$ is a separable Banach space. Show that both Y and X^* are separable.

3. Let X be a Banach space and let $A \subset X^*$ be a set separating points of X. Let τ be the smallest vector space topology on X making A continuous. Let $\wedge : (X, \tau) \to \mathbb{C}$ be a bounded linear map. Show that $\wedge \in$ span A.

4. Let $h : [0, 1] \to L^\infty[0, 1]$ be a continuous function. Define $T : L^1[0, 1] \to C[0, 1]$ by $T(f)(t) = \int_0^1 f(s)\ h(t)(s)\ ds$ where $t \in [0, 1]$ and $f \in L^1[0, 1]$. Show that T is well-defined and $\|T\| = \sup\limits_{t\in[0,1]} \|h(t)\|$.

5. Let A be a C^*-algebra of compact operators on a Hilbert space. Show that A is an ideal in its double commutant.

6. Show that the identity of a C^*-algebra, as well as any unitary vector is an extreme point of the closed unit ball.

7. Show that the positive cone of self adjoint elements in a C^*-algebra with identity is a closed set and the norm is monotone on this set, i.e., $0 \leq x \leq y \Rightarrow \|x\| \leq \|y\|$.

8. Let A be a C^*-algebra with identity and $x^* = x \in A$. Suppose $f(x) = 0$ for every state f. Show that $x = 0$, using functional calculus.

9. Let J be a proper ideal of a unital C^*-algebra A. Let $\pi : J \to B(H)$ be an irreducible representation. Show that π has a unique extension to an irreducible representation on A.

10. Let $\hat{A}$ be the C^*-algebra obtained by adjoining an identity to the C^*-algebra A. Let $f : A \to \mathbb{C}$ be a positive linear functional. Show that f has a positive extension to $\hat{A}$.